PART 1: QUESTIONS

Name:________________________________ **Age:**_____ **Id:**_______________ **Course:**__________

Integrals - Exam 2 **Lesson: 4-6**

Instructions:

- Please begin by printing your Name, your Age, your Student Id , and your Course Name in the box above and in the box on the solution sheet.
- You have 90 minutes (class period) for this exam.
- You can not use any calculator, computer, cellphone, or other assistance device on this exam. However, you can set our flag to ask permission to consult your own one two-sided-sheet notes at any point during the exam (You can write concepts, formulas, properties, and procedures, but questions and their solutions from books or previous exams are not allowed in your notes).
- Each multiple-choice question is worth 5 points and each extra essay-question is worth from 0 to 5 points. (Even a simple related formula can worth some points).
- Set up your flag if you have a question.
- Relax and use strategies to improve your performance.

Exam Strategies to get the best performance:

- Spend 5 minutes reading your exam. Use this time to classify each Question in (E) Easy, (M) Medium, and (D) Difficult.
- Be confident by solving the easy questions first then the medium questions.
- Be sure to check each solution. In average, you only need 30 seconds to test it. (Use good sense).
- Don't waste too much time on a question even if you know how to solve it. Instead, skip the question and put a circle around the problem number to work on it later. In average, the easy and medium questions take up half of the exam time.
- Solving the all of the easy and medium question will already guarantee a minimum grade. Now, you are much more confident and motivated to solve the difficult or skipped questions.
- Be patient and try not to leave the exam early. Use the remaining time to double check your solutions.

1. Given:

I. The Table of Integrals and Substitution are the first attempts to solve integrals.

II. The integration by parts is a powerful technique to solve a much larger set of functions.

III. The Column Method is a practical and an easy way to solve Integration by Parts.

a) Only I and II are correct.
b) Only I and III are correct.
c) Only II and III are correct.
d) I, II, and III are **incorrect**.
e) I, II, and III are correct.

Solution: e

The Table of Integrals and Substitution are the first attempts to solve integrals. The Integration by Parts is a powerful technique to solve a much larger set of functions normally used when Table of Integrals and Substitution are not an option. The Column Method is a practical and easy way to solve Integrals by Parts.

2. Let u and v be any real functions.

The formula of Integral by Parts is:

a) $\int u\,dv = -uv + \int v\,du.$

b) $\int u\,dv = uv + \int v\,du.$

c) $\int u\,dv = -uv - \int v\,du.$

d) $\int u\,dv = uv - \int v\,du.$

e) None of the above.

Solution: d

The Integration by Parts is a powerful technique to solve a much larger set of functions.

The main idea of this technique is to transform an integral $\int u\,dv$ in an another integral $\int v\,du$ using the following formula:

$$\int u\,dv = uv - \int v\,du,$$

where u and v are differential functions.

3. What is important about the Integration by Parts:

I. It is a powerful technique to solve a much larger set of functions.

II. To solve the integral $\int u\,dv$ using the Integration by Parts, students should to identify u and dv properly to converge to an easier Integral.

III. To solve the integral $\int u\,dv$, students should first to compute du and v to use in the Integral by Parts formula.

Then:

a) Only I and II are correct.
b) Only I and III are correct.
c) Only II and III are correct.
d) I, II, and III are correct.
e) None of the above.

Solution: d

The Integration by Parts $\left(\int u\,dv = uv - \int v\,du\right)$ is a powerful technique to solve a much larger set of functions. To solve the integral $\int u\,dv$, students should identify u and dv properly and compute du and v to converge to an easier Integral. Although the Standard Method solution is long and requires time, it is a common tool in exams. On the other hand, the Column Method is a practical and easier way to solve integrals by parts.

4. What is important to know about the integral by parts methodologies:

a) The Standard Method solution is easier than the Column Method to solve integrals by parts.
b) The Column Method is the most used in all universities around the world and the Standard Method will be no longer be used.
c) The Column Method solve complex integrals, but it requires a lot of time to be used in exams.
d) The standard Method formula to solve integral by parts could be applied several times to find an easier integral. In the other hand, the Column Method can be used as a guide to get faster and the same solution offered by the Standard Method in an organized and compact table.
e) The Column Method is prohibited to be used in certain universities because several professors don't knowing it.

Solution: d

The Standard Method is the most common and classical way to solve integral by parts. However, this method could be repeated several times to get an easier integral. On the other hand, the Column Method is an easy and organized way to solve integrals by parts. Even if your professor doesn't know the Column Method, you can used it in your exam as guide to get faster the same solution offered by the Standard Method.

5. To solve the integral $\int u\, dv$ by integration by parts effectively, students should to identify the function u properly.

Given the notation:

E-Exponential,
T-Trigonometric,
L-Logarithmic,
P-Polynomial.

A rule of thumb to choose smartly the function u is:

a) LPTE
b) LTPE
c) EPTL
d) ETPL
e) ETLP

Solution: a

A rule of thumb to choose smart the function u is LPTE (Let Patrick Teach Easy).

6. Solve the following integral:

$$I = \int_0^1 dx$$

a) $\frac{1}{5}$ b) $\frac{1}{4}$ c) $\frac{1}{3}$ d) $\frac{1}{2}$ e) 1

Solution: e

$$I = \int_0^1 dx = x\Big|_0^1 = (1) - (0) = 1.$$

7. Solve the following integral:

$$I = \int e^{x^4} x^3\, dx$$

a) $I = \frac{1}{2}e^{x^2} + c$
b) $I = \frac{1}{3}e^{x^3} + c$
c) $I = \frac{1}{4}e^{x^4} + c$
d) $I = \frac{1}{5}e^{x^5} + c$
e) None of the above.

Solution: c

Let $u = x^4$ and $du = 4x^3 dx$.

$$I = \int e^{x^4} 4x^3 \, dx = \frac{1}{4} \int e^{x^4} 4x^3 \, dx$$

Making the substitution:

$$I = \frac{1}{4} \int e^u \, du \Rightarrow I = \frac{1}{4} e^u + c$$

Thus, $I = \frac{1}{4} e^{x^4} + c$.

8. Solve the following integral:

$$I = \int e^x x^3 \, dx$$

a) $I = xe^x - e^x + c$
b) $I = -xe^x - e^x + c$
c) $I = x^2 e^x - 2xe^x + 2e^x + c$
d) $I = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$
e) None of the above.

Solution: d

$$I = \int e^x x^3 \, dx$$

Let $u = x^3$ and $v = e^x$.

The table of the Column Method is:

D		I
x^3	+	e^x
$3x^2$	-	e^x
$6x$	+	e^x
6	-	e^x
0	+	e^x

$$I = \int e^x x^3 \, dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + \int e^x .0 \, dx$$

Thus, $I = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$.

9. Solve the following integral:

$$I = \int x^3 \cos x \, dx$$

a) $I = -x \cos x + \sin x + c$
b) $I = x \sin x + \cos x + c$
c) $I = -x^2 \cos x + 2x \sin x + 2 \cos x + c$
d) $I = x^2 \sin x + 2x \cos x - 2 \sin x + c$
e) None of the above.

Solution: e

$$I = \int x^3 \cos x \, dx$$

Let $u = x^3$ and $v = \cos x$.

The table of the Column Method is:

D		I
x^3	+	$\cos x$
$3x^2$	-	$\sin x$
$6x$	+	$-\cos x$
	-	$-\sin x$
0		$\cos x$

$$I = \int x^3 \cos x \, dx$$

$$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + \int \cos x .0 \, dx$$

Thus,

$$I = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c.$$

10. Solve the following integral:

$$I = \int x^3 \ln x \, dx$$

a) $I = \dfrac{x^2}{2} \ln x - \dfrac{x^2}{4} + c$

b) $I = x \sin x + \cos x + c$

c) $I = -x^2 \cos x + 2x \sin x + 2 \cos x + c$

d) $I = x^2 \sin x + 2x \cos x - 2 \sin x + c$

e) None of the above.

Solution: c

$$I = \int x^3 \ln x \, dx$$

Let $u = x^3$ and $v = \ln x$.

The table of the Column Method is:

D		I
$\ln x$	$+$	x^3
$1/x$	$-$	$x^4/4$

$$I = \int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{1}{x} \cdot \frac{x^4}{4} \, dx$$

$$I = \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \, dx$$

$$I = \frac{x^4}{4} \ln x - \frac{x^4}{16}$$

Thus, $I = \dfrac{x^4}{4} \ln x - \dfrac{x^4}{16} + c.$

11. Solve the following integral:

$$I = \int \cos x e^{-x} dx$$

a) $I = \dfrac{\sin x e^{x}}{4} - \dfrac{\cos x e^{x}}{4} + c$

b) $I = -\dfrac{\sin x e^{-x}}{4} + \dfrac{\cos x e^{-x}}{4} + c$

c) $I = \dfrac{\cos x e^{x}}{4} + \dfrac{\sin x e^{x}}{4} + c$

d) $I = -\dfrac{\cos x e^{-x}}{4} + \dfrac{\sin x e^{-x}}{4} + c$

e) None of the above.

Solution: e

$$I = \int \cos x e^{-x} dx$$

Let $u = \cos x$ and $v = e^{-x}$.

The table of the Column Method is:

D		I
$\cos x$	$+$	e^{-x}
$-\sin x$	$-$	$-e^{-x}$
$-\cos x$	$+$	e^{-x}

$$I = \int \cos x e^{-x} dx = -\cos x e^{-x} + \sin x e^{-x} - \int \cos x e^{-x} dx$$

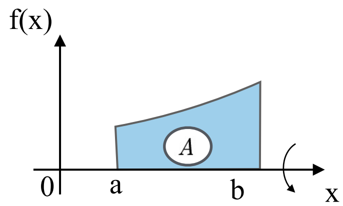
$$I = -\cos x e^{-x} + \sin x e^{-x} - I$$

$$2I = -\cos x e^{-x} + \sin x e^{-x}$$

$$I = \frac{-\cos x e^{-x}}{2} + \frac{\sin x e^{-x}}{2}$$

Thus, $I = -\dfrac{\cos x e^{-x}}{2} + \dfrac{\sin x e^{-x}}{2} + c.$

12. Rotating an area ($A \neq 0$) over the x− axis we have:

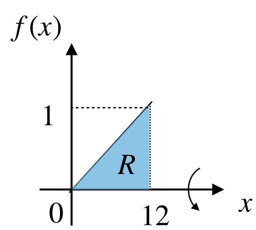


a) Point b) Line c) Area d) Volume e) Empty space.

Solution: d

The volume of the solid of revolution is obtained from revolving the Area (A) the graph $y = f(x)$ from $x = a$ to $x = b$ over the x - axis.

13. Find the volume (V) generated by rotating the following regions (R) over the x− axis.
Given $f(x) = \dfrac{x}{12}$.



a) π b) 2π c) 3π d) 4π e) None of the above.

Solution: d

$$V = \int_a^b \underbrace{\pi f(x)^2 dx}_{\text{dv}}$$

$$V = \int_0^{12} \pi\left(\frac{x}{12}\right)^2 dx$$

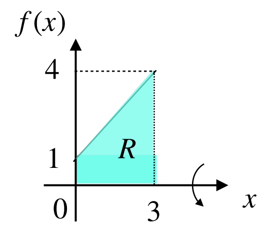
$$V = \frac{\pi}{144}\int_0^{12} x^2 dx$$

$$V = \frac{\pi}{144}\left[\frac{x^3}{3}\right]_0^{12}$$

$$V = \frac{\pi}{144}\left[\frac{12^3}{3} - \frac{0^3}{3}\right]$$

Thus, $V = 4\pi$.

14. Find the volume (V) generated by rotating the following region (R) over the x− axis.
Given $f(x) = x + 1$.



a) 21π b) 114π c) 333π
d) 732π e) None of the above.

Solution: a

$$V = \int_a^b \underbrace{\pi f(x)^2 dx}_{\text{dv}}$$

$$V = \int_0^3 \pi(x+1)^2 dx$$

Let $u = x + 1 \Rightarrow du = dx$

Note: $u = (0) + 1 = 1$ and $u = (3) + 1 = 4$

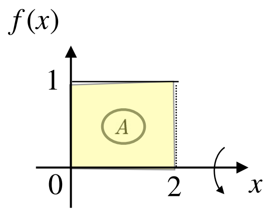
$$V = \pi\int_1^4 u^2 du$$

$$V = \pi\left[\frac{x^3}{3}\right]_1^4$$

$$V = \pi\left[\frac{4^3}{3} - \frac{1^3}{3}\right]$$

Thus, $V = 21\pi$.

15. Find the volume (V) generated by rotating the following area (A) over the x− axis. Given $f(x) = 1$.



a) π b) 2π c) 3π d) 4π e) None of the above.

Solution: b

$$V = \int_a^b \underbrace{\pi f(x)^2 dx}_{\text{dv}}$$

$$V = \int_0^2 \pi (1)^2 dx$$

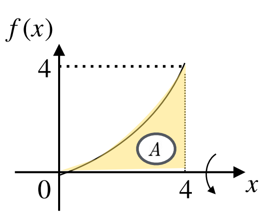
$$V = \pi \int_0^2 dx$$

$$V = \pi \Big[x\Big]_0^2$$

$$V = \pi\Big[(2) - (0)\Big]$$

Thus, $V = 2\pi$.

16. Find the volume (V) generated by rotating the following area (A) over the x− axis. Given $f(x) = x^2$.



a) $\dfrac{\pi}{5}$ b) $\dfrac{2^5\,\pi}{5}$ c) $\dfrac{3^5\,\pi}{5}$ d) $\dfrac{4^5\,\pi}{5}$ e) $5^4\,\pi$

Solution: d

$$V = \int_a^b \underbrace{\pi f(x)^2 dx}_{\text{dv}}$$

$$V = \int_0^4 \pi \big[x^2\big]^2 dx$$

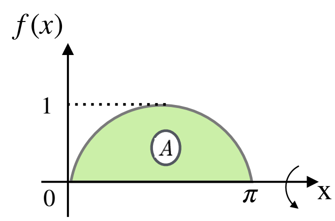
$$V = \pi \int_0^4 x^4\, dx$$

$$V = \pi \left[\frac{x^5}{5}\right]_0^4$$

$$V = \pi\left[\frac{(4)^5}{5} - \frac{(0)^5}{5}\right]$$

Thus, $V = \dfrac{4^5\pi}{5}$.

17. Find the volume (V) generated by rotating the following area (A) over the x− axis. Given $f(x) = \sin x$.



a) $\dfrac{\pi^2}{2}$ b) π^2 c) $\dfrac{3\pi^2}{2}$ d) $2\pi^2$ e) $4\pi^2$.

Solution: a

$$V = \int_0^\pi \pi \sin x \sin x\, dx$$

$$V = \pi \int_0^\pi \sin x \sin x\, dx$$

Let $I = \displaystyle\int \sin x \sin x\, dx = \int \sin^2 x\, dx.$

D		I
$\sin x$	$+$	$\sin x$
$\cos x$	$-$	$-\cos x$

$I = \sin x \cos x + \int \cos^2 x \, dx$

$I = \sin x \cos x + \int 1 \, dx - \int \sin^2 x \, dx$

$I = \sin x \cos x + x - I$

$2I = \sin x \cos x + x$

$I = \dfrac{\sin x \cos x}{2} + \dfrac{x}{2} + c$

Then, $V = \pi \left[\dfrac{\sin x \cos x}{2} + \dfrac{x}{2} \right]_0^{\pi}$

$V = \pi \left[\left(\dfrac{\sin(\pi)\cos(\pi)}{2} + \dfrac{(\pi)}{2} \right) - \left(\dfrac{\sin(0)\cos(0)}{2} + \dfrac{(0)}{2} \right) \right]$

$V = \dfrac{\pi^2}{2}.$

18. Find $I = \displaystyle\int_0^1 e^{\ln x} dx.$

a) $\dfrac{1}{4}$ b) $\dfrac{1}{3}$ c) $\dfrac{1}{2}$ d) 2 e) None of the above.

Solution: c

Note: $e^{\ln x} = x.$

$I = \displaystyle\int_0^1 e^{\ln x} dx = \int_0^1 x \, dx = \left. \frac{x^2}{2} \right|_0^1 = \left[\frac{(1)^2}{2} - \frac{(0)^2}{2} \right] = \frac{1}{2}$

Thus, $I = \dfrac{1}{2}.$

19. Find $I = \displaystyle\int_0^{\frac{\pi}{4}} 2 \sin x \cos x \, dx.$

a) $\dfrac{1}{6}$ b) $\dfrac{1}{5}$ c) $\dfrac{1}{4}$ d) $\dfrac{1}{3}$ e) None of the above.

Solution: e

Note: $2 \sin x \cos x = \sin(2x).$

$I = \displaystyle\int_0^{\frac{\pi}{4}} 2 \sin x \cos x \, dx = \int_0^{\frac{\pi}{4}} \sin(2x) \, dx.$

Let $u = 2x \Rightarrow du = 2dx$

Note: $u = 2(0) = 0$ and $u = 2\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{2}$

$I = \dfrac{1}{2} \displaystyle\int_0^{\frac{\pi}{2}} \sin u \, du$

$I = -\dfrac{1}{2} \left[\cos u \right]_0^{\frac{\pi}{2}}$

$I = -\dfrac{1}{2} \left[\cos\left(\dfrac{\pi}{2}\right) - \cos(0) \right] = \dfrac{1}{2}$

Thus, $I = \dfrac{1}{2}.$

20. The group of functions whose their derivatives is the same that their Integrals is:

a) $f(x) = Ce^x$, where $C \in \mathbb{R}$.

b) $f(x) = C \ln x$, where $C \in \mathbb{R}$.

c) $f(x) = Cx$, where $C \in \mathbb{R}$.

d) $f(x) = C \sin x$, where $C \in \mathbb{R}$.

e) None of the above.

Solution: a

$f(x) = Ce^x \Rightarrow f'(x) = Ce^x$

Then $\int f'(x)dx = f(x) + c = Ce^x + c$

For $c = 0 \Rightarrow I = Ce^x$.

PART 2: SOLUTIONS

Consulting

Name:______________________________ **Age:**_____ **Id:**______________ **Course:**__________

Multiple-Choice Answers

Questions	A	B	C	D	E
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
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16					
17					
18					
19					
20					

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	Points	Max
Multiple Choice		100
Extra Points		25
Consulting		10
Age Points		25
Total Performance		160
Grade		A

Extra Questions

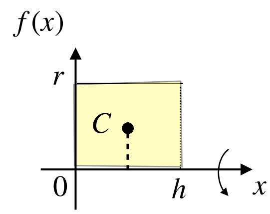
21. The famous theorem of Pappus states that the volume V of a solid of revolution generated by the revolution of a area A over the x axis is equal to:

$$V = 2\pi d A,$$

where d is the distance between x-axis and the centroid of the area A.

Show that the area for a cylinder is $V = \pi r^2 h$.

Hint: $d = \dfrac{r}{2}$.



Solution:

Let $d = \dfrac{r}{2}$ and $A = h\,.\,r$, then:

By Pappus' s theorem, we have:

$V = 2\pi d A \Rightarrow V = 2\pi\left(\dfrac{r}{2}\right)h\,.\,r$

Thus, $V = \pi r^2 h$.

22. Show that:

$$I = \int \ln x \, dx = \ln x - x + c, \text{ where } x > 0.$$

Solution:

$$I = \int \ln x \, dx$$

D		I
$\ln x$	$+$	1
$1/x$	$-$	x

$$I = \ln x - \int x \, . \frac{1}{x} \, dx$$

$$I = \ln x - \int dx$$

Thus, $I = \ln x - x + c$.

23. Find $I = \int x^2 \, . \, x \, dx$.

a) Standard Method (5 points)

b) Column Method (5 points)

Solution:

a) Standard Method

$$I = \int x^2 \, . \, x \, dx$$

Let $u = x^2 \Rightarrow du = 2dx$

$$dv = x du \Rightarrow v = \int x dx \Rightarrow v = \frac{x^2}{2}$$

$$\int u dv = uv - \int v du$$

$$I = \int x^2 \, . \, x \, dx = x^2 \, . \, \frac{x^2}{2} - \int \frac{x^2}{2} . 2x dx$$

$$I = \frac{x^4}{2} - I \Rightarrow 2I = \frac{x^4}{2} \Rightarrow I = \frac{x^4}{4}$$

Thus, $I = \frac{x^4}{4} + c$.

b) Column Method

$$I = \int x^2 \, . \, x \, dx$$

D		I
x^2	$+$	x
$2x$	$-$	$x^2/2$

$$I = x^2 \, . \, \frac{x^2}{2} - \int \frac{x^2}{2} . 2x dx$$

$$I = \frac{x^4}{2} - I \Rightarrow 2I = \frac{x^4}{2} \Rightarrow I = \frac{x^4}{4}$$

Thus, $I = \frac{x^4}{4} + c$.

24. Find $I = \displaystyle\int_e^{\pi} x\,dx$.

Solution: $I = \dfrac{\pi^2 - e^2}{2}$

$$I = \int_e^{\pi} x\,dx = \frac{x^2}{2}\bigg|_e^{\pi} = \frac{\pi^2 - e^2}{2}$$

Thus, $I = \dfrac{\pi^2 - e^2}{2}$.

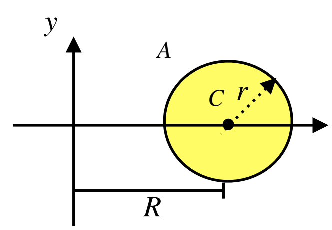
25. **Pappus's formula challenge.**

The famous theorem of Pappus states that the volume V of a solid of revolution generated by the revolution of a area A over the x axis is equal to:

$$V = 2\pi d A,$$

where d is the distance between x-axis and the centroid of the area A.

Find the Torus volume by rotating a circle (radius $= r$) over y-axis. Given the distance between the center and the y-axis is R.



Note: If you draw a happy face, you will receive anyway an extra 5 points.

Solution: $V = 2\pi^2 R r^2$

$$V = 2\pi d A \Rightarrow V = 2\pi R(\pi r^2)$$

Thus, $V = 2\pi^2 R r^2$.